

# The Dirichlet Casimir effect for $\phi^4$ theory in $(3 + 1)$ dimensions: a new renormalization approach

---

**Reza Moazzemi, Maryam Namdar and Siamak S. Gousheh**

*Department of Physics, Shahid Beheshti University,  
Evin, Tehran, 19839, Iran*

*E-mail:* R-Moazzemi@sbu.ac.ir, m-namdar@sbu.ac.ir, ss-gousheh@sbu.ac.ir

**ABSTRACT:** We calculate the next to the leading order Casimir effect for a real scalar field, within  $\phi^4$  theory, confined between two parallel plates in three spatial dimensions with the Dirichlet boundary condition. In this paper we introduce a systematic perturbation expansion in which the counterterms automatically turn out to be consistent with the boundary conditions. This will inevitably lead to nontrivial position dependence for physical quantities, as a manifestation of the breaking of the translational invariance. This is in contrast to the usual usage of the counterterms in problems with nontrivial boundary conditions, which are either completely derived from the free cases or at most supplemented with the addition of counterterms only at the boundaries. Our results for the massive and massless cases are different from those reported elsewhere. Secondly, and probably less importantly, we use a supplementary renormalization procedure, which makes the usage of any analytic continuation techniques unnecessary.

**KEYWORDS:** Renormalization Regularization and Renormalons, Field Theories in Higher Dimensions.


---

## Contents

<b>1. Introduction</b>	<b>1</b>
<b>2. The leading term of the Casimir effect</b>	<b>4</b>
<b>3. First-order radiative correction</b>	<b>6</b>
3.1 Renormalization conditions	7
3.2 The massive case	9
3.3 The massless case	11
<b>4. On the analytic continuation techniques</b>	<b>12</b>
<b>5. Conclusions</b>	<b>13</b>
<b>A. Use of the Abel-Plana summation formula in our subtraction scheme</b>	<b>14</b>

---

## 1. Introduction

During the last fifty years many papers have been written on the Casimir effect. In this paper we introduce a new approach in regards to the renormalization program. We use our approach in one of the simplest nontrivial example possible, i.e. a real scalar field confined between two parallel plates in 3+1 dimensions, with  $\phi^4$  self-interaction. As we shall see, our results for the next to leading order term (NLO) differs significantly from what exists in the literature. It is therefore suitable to start at the beginning. In 1948 H.B.G. Casimir found a simple yet profound explanation for the retarded van der Waals interaction [1]. After a short time, he and D. Polder related this effect to the change in the zero point energy of the quantum fields due to the presence of nontrivial boundary conditions [2]. This energy has since been called the Casimir energy. The zero-order energy in perturbation theory has been calculated for various fields (see for example [3, 4]). Also the NLO correction, which is usually called the first-order effect, has been computed for various fields. For the electromagnetic field this correction is said to be due to the following Feynman diagram , and has been computed first by Bordag and collaborators [5]. However, note that this correction is a two loop correction in this case and is  $\mathcal{O}(e^2)$ . Moreover the two-loop radiative corrections for some effective field theories has been investigated in [6–8]. Next, in the case of a real massive scalar field NLO correction to the energy has been computed in [9–16]. This correction is a two loop correction in this case but is  $\mathcal{O}(\lambda)$ . Moreover, N. Graham *et al* used new approaches to this problem by utilizing the phase shift of the scattering states [17], or replacing the boundary conditions by an appropriate potential

term [18]. However, all of the authors who use counterterms to calculate NLO correction to the Casimir energy use the free counterterms (in the space between the plates) by which we mean the ones that are relevant to the free cases with no nontrivial boundary conditions, and are obviously position independent. Only in [13] the author notes that in certain cases, counterterms can depend on the distance between the plates. The first use of nontrivial boundary conditions for the renormalization programs in problems of this sort seems to be due to Fosco and Svaiter [19]. These authors use free counterterms in the space between the plates and place additional surface counterterms at the boundaries. Later on various authors proposed the use of exactly the same renormalization procedure for various physical problems [20]. The first calculation for the NLO of Casimir energy for the massive scalar field using this renormalization program is done in ref. [21]. We should note that their results for the massless limit in 1+1 dimensions is infinite similar to [15], who used free counterterms only. Moreover, the results for the massless limit in 2 + 1 case reported in ref. [15, 21], depend crucially on the order in which the limits  $d \rightarrow 2$  and  $m \rightarrow 0$  are taken. When the order of the limits are as shown, their result is infinite. When the order is opposite, they get finite results which, surprisingly, contains the Euler-Mascheroni number for both the massive and massless cases. This is in contrast to the usual point of view that this number should not appear in any results which reflect a physical quantity [22, 23]. We interpret these negative results as indications that the use of free counterterms in these problems might not be appropriate. It is also worth mentioning that all the papers on the analogous calculations of the NLO corrections to the mass of solitons, that we are aware of, use free counterterms (see for example [24–26]). In references [25] the authors used the mode number cutoff introduced by R. F. Dashen (1974) [26] to calculate the NLO Casimir energy due to the presence of solitons.

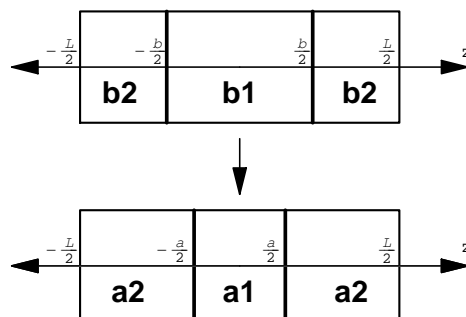
In this paper, we present a systematic approach to the renormalization program for problems which are amenable to renormalized perturbation theory, and contain either nontrivial boundary conditions or nontrivial (position dependent) backgrounds, e.g. solitons, or both. Obviously all the n-point functions of the theory will have in general nontrivial position dependence in the coordinate representation. This is one of the manifestations of the breaking of the translational symmetry. The procedure to deduce the counterterms from the n-point functions in a renormalized perturbation theory is standard and has been available for over half a century. Using this, as we shall show, we will inevitably obtain position dependent counterterms. Therefore, the radiative corrections to all the input parameters of the theory, including the mass, will be in general position dependent. Therefore, we believe the information about the nontrivial boundary conditions or position dependent backgrounds are carried by the full set of n-point functions, the resulting counterterms, and the renormalized parameters of the theory. Our preliminary investigations have revealed that the main difference between our position dependent counterterms and the free ones are maximal for positions which are about a Compton wavelength away from the walls, although it is also nontrivial at other places. Moreover, in the limit of large plate separation our counterterms become position independent everywhere except within one Compton wavelength away from the walls. In this limit, the difference between our counterterms and the free ones, before the transverse momentum integration, approaches

a constant value proportional to  $1/2m$  away from the walls and  $1/m$  at the walls. However since these terms will multiply at least one-loop expressions which are divergent, their difference will have significant consequences. Here we use this procedure to compute the first-order radiation correction to the Casimir energy for a real scalar field in 3+1 dimensions with  $\phi^4$  self interaction. We compute this correction for both a massive and a massless scalar fields and show that the massless limit of the massive case exactly corresponds to the massless case.

In addition, up to now, most of the papers on the Casimir effect that we are aware of, use some form of *analytic continuation*. We share the point of view with some authors such as the ones in [10, 14] that the analytic continuation techniques are not always completely justified physically. Moreover, like the authors of the first of the aforementioned references, we have found counterexamples, which we point out in this paper and elsewhere [27, 28]. The counterexamples show that analytic continuation techniques alone might not yield correct physical results, and sometimes even give infinite results [29]. Therefore, we prefer to use a completely physical approach by enclosing the whole system in a box of volume  $V = L^3$ , which eventually can go to infinity, and calculating the difference between the zero point energies of two different configurations. The main idea of this method is actually due to T. H. Boyer [30], who used spheres instead of boxes. This we shall call the “box renormalization scheme” and can be used as a supplementary part of other usual regularization or renormalization programs. This box renormalization scheme, has the following advantages:

- (i) Use of this procedure removes all of the ambiguities associated with the appearance of the infinities, and we use the usual prescription for removing the infinities in the regulated theory, as explained in section 3.2. This is all done without resorting to any analytic continuation schemes.
- (ii) In order to calculate the Casimir energy we subtract two physical configuration of similar nature, e.g. both confined within finite regions, and not one confined and the other in an unbounded region.
- (iii) This method can be used as a check for the cases where analytic continuation yields finite results, and more importantly, can be used to obtain finite results when the former yields infinite results.

More importantly, we have discovered for the case of parallel plates, much to our surprise, that when the problem is set up correctly, that is when proper counterterms are used, using analytic continuation technique for NLO correction it seems impossible to obtain finite results in any integer space-time dimensions, and correct finite results in any non-integer space-time dimensions. We will make this statement more explicit later on in section 4. However, this technique also gives the correct results for the leading term for this geometry (see for example [3]). We should mention that some authors believe that use of box regularization or renormalization procedures, in which the size of the box eventually goes to infinities could be avoided by using appropriate boundary conditions on the fields at spatial infinity [31].



**Figure 1:** The geometry of the two different configurations whose energies are to be compared. The labels  $a_1$ , *etc.* denote the appropriate sections in each configuration separated by the plates.

In section 2 we calculate the leading order term for the Casimir energy in  $d$  space dimensional case. We do this first of all to explain more completely the physical content of the problem and set up our notations. Secondly this computation is just about as easy to do in  $d$  dimensions as is in the three dimensional case. In section 3 we compute the first order radiative correction to this energy. In order to do this we first state the renormalization condition, and then derive an expression for the first order radiative correction for a massive and massless scalar field. In section 4 we discuss the validity of the analytic continuation techniques relevant to our problem. In section five we give a brief summary of our results and state our conclusions.

## 2. The leading term of the Casimir effect

The lagrangian density for a real scalar field with  $\phi^4$  self-interaction is:

$$\mathcal{L}(x) = \frac{1}{2}[\partial_\mu\varphi(x)]^2 - \frac{1}{2}m_0^2\varphi(x)^2 - \frac{\lambda_0}{4!}\varphi(x)^4, \tag{2.1}$$

where  $m_0$  and  $\lambda_0$  are the bare mass and bare coupling constant, respectively. Here we calculate the leading term for the Casimir energy in  $d$  spatial dimensions. Obviously the leading term, in contrast to the higher order corrections, is independent of the form of the self-interaction. The Casimir energy is in general equivalent to the work done on the system for bringing two parallel plates from  $\pm\infty$  to  $\pm a/2$ . As mentioned before, part of our renormalization procedure is to enclose the whole system in a  $d$  dimensional cubical box of sides  $L$ . To compute this leading term, we first compare the energies in two different configurations: when the plates are at  $\pm a/2$  as compare to  $\pm b/2$ . We name the axis perpendicular to the plates the  $z$  axis. To keep the expressions symmetrical, we choose the coordinates so that the edges of the confining box are at  $\pm L/2$  in any direction.

The total zero point energy of the upper configuration in figure (1) will be called  $E_b$  and of the lower one  $E_a$ . In our box renormalization scheme we need to define the Casimir energy as follows

$$E_{Cas.} = \lim_{b/a \rightarrow \infty} \left[ \lim_{L/b \rightarrow \infty} (E_a - E_b) \right], \tag{2.2}$$

where,

$$E_a = E_{a_1} + 2E_{a_2}, \quad E_b = E_{b_1} + 2E_{b_2}. \quad (2.3)$$

Here we choose the Dirichlet boundary condition on all of the boundaries. Then we can expand the field operator  $\varphi$  in the eigenstate basis appropriate to this boundary condition, and its explicit second quantized form, for example in region  $a_1$  becomes

$$\begin{aligned} \varphi_{a_1}(x) = \int \frac{d^{d-1}\mathbf{k}^\perp}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \left( \frac{1}{a\omega_{a_1,n}} \right)^{1/2} \left\{ e^{-i(\omega_{a_1,n}t - \mathbf{k}^\perp \cdot \mathbf{x}^\perp)} \sin \left[ k_{a_1,n} \left( z + \frac{a}{2} \right) \right] \mathbf{a}_n \right. \\ \left. + e^{i(\omega_{a_1,n}t - \mathbf{k}^\perp \cdot \mathbf{x}^\perp)} \sin \left[ k_{a_1,n} \left( z + \frac{a}{2} \right) \right] \mathbf{a}_n^\dagger \right\}, \end{aligned} \quad (2.4)$$

where,

$$\omega_{a_1,n}^2 = m_0^2 + k^{\perp 2} + k_{a_1,n}^2, \quad k_{a_1,n} = \frac{n\pi}{a} \quad \text{and} \quad n = 1, 2, \dots, \quad (2.5)$$

here  $\mathbf{k}^\perp$  and  $k_{a_1,n}$  denote the momenta parallel and perpendicular to plates, respectively. Also  $\mathbf{a}_n^\dagger, \mathbf{a}_n$  are creation and annihilation operators obeying the usual commutation relations:

$$[\mathbf{a}_n, \mathbf{a}_{n'}^\dagger] = \delta_{n,n'}, \quad [\mathbf{a}_n, \mathbf{a}_{n'}] = [\mathbf{a}_n^\dagger, \mathbf{a}_{n'}^\dagger] = 0,$$

and  $\mathbf{a}|0\rangle = 0$  defines the vacuum state in the presence of boundary conditions. Using the above equations one can easily obtain

$$\begin{aligned} E_{a_1}^{(0)} = \int d^d\mathbf{x} \langle 0 | \mathcal{H}^{(0)} | 0 \rangle = L^{d-1} \int \frac{d^{d-1}\mathbf{k}^\perp}{(2\pi)^{d-1}} \sum_{n=1}^{\infty} \frac{\omega_{a_1,n}}{2} \\ = \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^\infty dk k^{d-2} \sum_{n=1}^{\infty} \omega_{a_1,n}, \end{aligned} \quad (2.6)$$

where  $\mathcal{H}^{(0)}$  denotes the usual free Hamiltonian density, easily obtained from the Lagrangian density, and the superscript (0) denotes the zero (or leading) order term of this energy. Also  $k = |\mathbf{k}^\perp|$ , and  $\Omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}$  is the solid angle in  $d$ -dimensions. Therefore,

$$E_a^{(0)} - E_b^{(0)} = \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^\infty dk k^{d-2} \sum_n g(n), \quad (2.7)$$

where,

$$g(n) = \omega_{a_1,n} + 2\omega_{a_2,n} - \omega_{b_1,n} - 2\omega_{b_2,n}.$$

Now we are allowed to use the Abel-Plana summation formula, since we now expect the summand to satisfy the strict conditions [36] for the validity of this formula. That is, we expect any reasonable renormalization program for calculating any measurable physical quantity to yield finite results. The Abel-Plana summation formula gives

$$E_a^{(0)} - E_b^{(0)} = \frac{L^{d-1}}{2} \frac{\Omega_{d-1}}{(2\pi)^{d-1}} \int_0^\infty dk k^{d-2} \left[ \frac{-g(0)}{2} + \int_0^\infty g(x) dx + i \int_0^\infty \frac{g(it) - g(-it)}{e^{2\pi t} - 1} dt \right], \quad (2.8)$$

where  $g(0)$  vanishes in this case due to our box renormalization. The second term in the bracket, using suitable changes of variables, becomes

$$\frac{a}{\pi} \int_0^\infty d\kappa (m_0^2 + k^2 + \kappa^2)^{1/2} + 2 \frac{L-a}{2\pi} \int_0^\infty d\kappa (m_0^2 + k^2 + \kappa^2)^{1/2} \quad (2.9)$$

$$- \frac{b}{\pi} \int_0^\infty d\kappa (m_0^2 + k^2 + \kappa^2)^{1/2} - 2 \frac{L-b}{2\pi} \int_0^\infty d\kappa (m_0^2 + k^2 + \kappa^2)^{1/2} = 0,$$

where  $\kappa$  for example in the first term denotes  $\frac{n\pi}{a}$  treated as a continuous variable. The above calculation shows that this term is exactly zero. Therefore, only the branch-cut term (the last term in eq. (2.8)) gives nonzero contribution and the final result is

$$E_a^{(0)} - E_b^{(0)} = - \frac{2L^{d-1} m_0^{(d+1)/2}}{(4\pi)^{(d+1)/2}} \sum_{j=1}^\infty \frac{1}{j^{(d+1)/2}} \left\{ \frac{K_{(d+1)/2}(2ajm_0)}{a^{(d-1)/2}} - \frac{K_{(d+1)/2}(2bjm_0)}{b^{(d-1)/2}} \right. \quad (2.10)$$

$$\left. + \frac{2K_{(d+1)/2}[(L-a)jm_0]}{\left(\frac{L-a}{2}\right)^{(d-1)/2}} - \frac{2K_{(d+1)/2}[(L-b)jm_0]}{\left(\frac{L-b}{2}\right)^{(d-1)/2}} \right\},$$

where  $K_n(x)$  denotes the modified Bessel function of order  $n$ . Using eq. (2.2) for the Casimir energy and noting that  $K_n(x)$  is strongly damped as  $x$  goes to infinity, only the first term remains when the limits are taken, and the result is

$$E_{\text{Cas.}}^{(0)} = - \frac{2L^{d-1}}{(4\pi)^{(d+1)/2}} \frac{m_0^{(d+1)/2}}{a^{(d-1)/2}} \sum_{j=1}^\infty \frac{K_{(d+1)/2}(2ajm_0)}{j^{(d+1)/2}}. \quad (2.11)$$

This is the result for the leading term for the Casimir energy in  $d$ -dimensions, on which all authors agree (see for example [3, 4]). It is important to note that we, unlike most other authors, derived this result without any use of analytic continuation. If we set  $d = 3$ , we have

$$E_{\text{Cas.}}^{(0)} = - \frac{L^2 m_0^2}{8\pi^2 a} \sum_{j=1}^\infty \frac{K_2(2ajm_0)}{j^2}, \quad (2.12)$$

with the following limits,

$$E_{\text{Cas.}}^{(0)} \rightarrow \begin{cases} \frac{-L^2}{8\pi^2 a} \sum_j \frac{1}{2a^2 j^4} = \frac{-L^2 \pi^2}{1440 a^3} & \text{as } m_0 \rightarrow 0, \\ \frac{-L^2}{8\sqrt{2}} \left(\frac{m_0}{\pi a}\right)^{3/2} e^{-2am_0} & \text{as } am_0 \gg 1. \end{cases} \quad (2.13)$$

The results are in agreement with what exists in literature (see for instance [4, 14]). It is interesting to note that for the massless case, the result is, not surprisingly, exactly half of the corresponding expression for the electromagnetic case.

### 3. First-order radiative correction

Now we calculate the next to the leading order (two loop quantum correction) shift of the Casimir energy for a scalar field in  $\phi^4$  theory using the renormalized perturbation theory

in 3 + 1 dimensions. As mentioned before, the main idea of our work is that when a systematic treatment of the renormalization program is done, the counterterms needed to retain the renormalization conditions, automatically turn out to be position dependent. This, as we shall see, will have profound consequences. However, our main scheme of canceling the divergences using counterterms and a few input experimental parameters, is in complete conformity with the standard renormalization approach. To set the stage for the calculations, we shall very briefly state the renormalization procedure and conditions.

### 3.1 Renormalization conditions

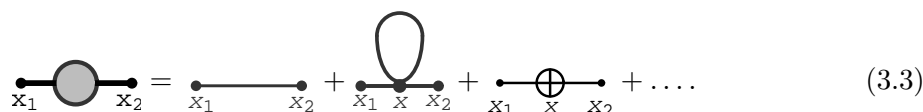
The  $\phi^4$  Lagrangian eq. (2.1), after rescaling the field  $\varphi = Z^{1/2}\varphi_r$ , where  $Z$  is called the field strength renormalization, and the standard procedure for setting up the renormalized perturbation theory, becomes (see for example [23]),

$$\mathcal{L}(x) = \frac{1}{2}[\partial_\mu\varphi_r(x)]^2 - \frac{1}{2}m^2\varphi_r(x)^2 - \frac{\lambda}{4!}\varphi_r(x)^4 + \frac{1}{2}\delta_Z[\partial_\mu\varphi_r(x)]^2 - \frac{1}{2}\delta_m\varphi_r(x)^2 - \frac{\delta_\lambda}{4!}\varphi_r(x)^4, \tag{3.1}$$

where  $\delta_m, \delta_\lambda, \delta_Z$  are the counterterms, and  $m$  and  $\lambda$  are the physical mass and physical coupling constant, respectively. In this problem we are to impose boundary conditions on the field at the walls. An alternative approach would be to add appropriate external potentials to the Lagrangian so as to maintain the boundary conditions on the fields [18]. We will use the first approach. Obviously the presence of nontrivial boundary conditions breaks the translational invariance and hence momenta will no longer be good quantum numbers. Therefore we find it easier to impose the renormalization conditions in the configuration space. For example, the standard expression for the two-point function is,

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle = \lim_{T\rightarrow\infty(1-i\epsilon)} \frac{\langle 0|\int \mathcal{D}\phi\phi(x_1)\phi(x_2)e^{i\int_{-T}^T \mathcal{L}d^4x}|0\rangle}{\langle 0|\int \mathcal{D}\phi e^{i\int_{-T}^T \mathcal{L}d^4x}|0\rangle}. \tag{3.2}$$

Since the birth of quantum field theory, as far as we know, the assertion has always been that the above expressions can be expanded systematically when the problem is amenable to perturbation theory. For example, in the context of renormalized perturbation theory, as indicated in eq. (3.1), we can symbolically represent the first few terms of the perturbation expansion of eq. (3.2) by



where  $\text{---}\oplus\text{---}$  refers to the appropriate counterterm. It is obvious that the above expression represents a systematic perturbation expansion, and most importantly, all of the propagators on the right hand side should be the one appropriate to the problem under consideration, that is they should have the same overall functional form as the first term. Our first renormalization condition is that the renormalized mass  $m$  should be the pole of



the propagator represented by the first term in (3.3). This implies the second and third diagrams should cancel each other out in the lowest order, and this in turn implies the cancelation of the UV divergences in that order, and that the counterterms will in general turn out to be position dependent. The renormalized mass  $m$  will then naturally turn out to be position dependent as well. However, we only need to fix the value of  $m(x)$  at one position between the plates by our renormalization condition. The exact functional dependence of  $m(x)$  will then be completely determined by the theory. That is, we insist the overall structure of the renormalization conditions such as above, and the counterterms appearing in them should be determined solely from within the theory, and not for example be imported from the free case. The equations are self deterministic and there is no need to take such actions. Obviously we still need a few experimental input parameters for the complete renormalization program, such as  $m(x)$  for some  $x$ . Analogous expression and reasonings could be easily stated for the four-point function.

To one-loop order, the standard renormalization conditions applied to eq. (3.3) and its four-point counterpart, give

$$\delta_Z(x) = 0, \quad \delta_m(x) = \frac{-i}{2} \frac{\text{loop}}{\bar{x}} = \frac{-\lambda}{2} G(x, x); \quad \text{and} \quad \delta_\lambda(x) = 0, \quad (3.4)$$

respectively. Here  $G(x, x')$  is the propagator of the real scalar field and  $x = (t, \mathbf{x})$ . Obviously the counterterms automatically incorporate the boundary conditions and are position dependent, due to the dependence of the two and four-point functions on such quantities. Now, the higher order contributions to the vacuum energy in the interval a1 (i.e.  $z \in [\frac{-a}{2}, \frac{a}{2}]$ ) is

$$\begin{aligned} \Delta E_{a_1} &= E_{a_1}^{(1)} + E_{a_1}^{(2)} + \dots = \int_V d^3 \mathbf{x} \langle \Omega | \mathcal{H}_I | \Omega \rangle \\ &= i \int_V d^3 \mathbf{x} \left( \frac{1}{8} \text{loop} + \frac{1}{2} \text{loop} + \frac{1}{8} \text{loop} + \dots \right), \end{aligned} \quad (3.5)$$

where  $\text{loop}$  with a cross =  $-i\delta_\lambda(x)$  and  $\text{loop}$  with a circle =  $i[p^2\delta_Z(x) - \delta_m(x)]$  refer to the counterterms. Accordingly, the  $\mathcal{O}(\lambda)$  contribution to the vacuum energy is

$$\begin{aligned} E_{a_1}^{(1)} &= i \int_V d^3 \mathbf{x} \left( \frac{1}{8} \text{loop} + \frac{1}{2} \text{loop} \right) \\ &= i \int_V d^3 \mathbf{x} \left[ \frac{-i\lambda}{8} G_{a_1}^2(x, x) - \frac{i}{2} \delta_m(x) G_{a_1}(x, x) \right], \end{aligned} \quad (3.6)$$

where  $G_{a_1}(x, x')$  is the propagator of the real scalar field in region a1. Using eqs. (3.4) and (3.6), we obtain,

$$E_{a_1}^{(1)} = \frac{-\lambda}{8} \int_V G_{a_1}^2(x, x) d^3 \mathbf{x}. \quad (3.7)$$

### 3.2 The massive case

As mentioned before, here we choose the Dirichlet boundary condition on the plates. Then, after the usual wick rotation, the expression for the Green's function in the four dimensional Euclidean space becomes

$$G_{a_1}(x, x') = \frac{2}{a} \int \frac{d^3k}{(2\pi)^3} \sum_n \frac{e^{-\omega(t-t')} e^{-i\mathbf{k}^\perp \cdot (\mathbf{x}^\perp - \mathbf{x}'^\perp)} \sin \left[ k_{a_1, n} \left( z + \frac{a}{2} \right) \right] \sin \left[ k_{a_1, n} \left( z' + \frac{a}{2} \right) \right]}{k^2 + k_{a_1, n}^2 + m^2 + i\epsilon}. \quad (3.8)$$

It is very important to note that the  $\delta_m(x) = \frac{-\lambda}{2} G(x, x)$  (c.f. eq. (3.4)) is explicitly position dependent. Using eq. (3.8) and eq. (3.7) and carrying out the spatial integration one obtains,

$$E_{a_1}^{(1)} = -\frac{\lambda L^2}{32\pi^4 a} \left[ \sum_{n, n'} \int_0^\infty dk k^2 \frac{1}{k^2 + k_{a_1, n}^2 + m^2} \int_0^\infty dk k^2 \frac{1}{k^2 + k_{a_1, n'}^2 + m^2} + \frac{1}{2} \sum_n \left( \int_0^\infty dk k^2 \frac{1}{k^2 + k_{a_1, n}^2 + m^2} \right)^2 \right]. \quad (3.9)$$

To compute the integrals we regulate them by a momentum cutoff  $\Lambda$ . Then we have

$$E_{a_1}^{(1)} = -\frac{\lambda L^2}{32\pi^4 a} \lim_{\Lambda_{a_1, n} \rightarrow \infty} \left[ \sum_{n, n'} \left( \Lambda_{a_1, n}^2 - \pi \omega'_{a_1, n} \Lambda_{a_1, n} + \frac{\pi^2}{4} \omega'_{a_1, n} \omega'_{a_1, n'} \right) + \frac{1}{2} \sum_n \left( \Lambda_{a_1, n}^2 - \pi \omega'_{a_1, n} \Lambda_{a_1, n} + \frac{\pi^2}{4} \omega'_{a_1, n}{}^2 \right) \right], \quad (3.10)$$

where  $\omega'_{a_1, n}{}^2 = k_{a_1, n}^2 + m^2$ . Now the NLO for the Casimir energy, eq. (2.2), has four terms the first one of which is shown above. It is clear that we can adjust the cutoffs for each region so that the quadratic divergences coming from different regions cancel each other. Similar cancelation occurs for the linear divergences. The cancelation of these divergent quantities without any residual finite terms is the usual practice in regulated theories [14], and this is the prescription that we shall use. We shall comment on this further when we use it one more time for eq. (3.12) below. Hence we get

$$E_a^{(1)} - E_b^{(1)} = -\frac{\lambda L^2}{128\pi^2} \sum_n \left[ \sum_{n'} \left( \frac{\omega'_{a_1, n} \omega'_{a_1, n'}}{a} - \frac{\omega'_{b_1, n} \omega'_{b_1, n'}}{b} + 4 \frac{\omega'_{a_2, n} \omega'_{a_2, n'}}{L-a} - 4 \frac{\omega'_{b_2, n} \omega'_{b_2, n'}}{L-b} \right) + \frac{1}{2} \left( \frac{\omega'_{a_1, n}{}^2}{a} - \frac{\omega'_{b_1, n}{}^2}{b} + 4 \frac{\omega'_{a_2, n}{}^2}{L-a} - 4 \frac{\omega'_{b_2, n}{}^2}{L-b} \right) \right]. \quad (3.11)$$

This computation is obviously complicated and plagued with a multitude of infinities. As explained before using the usual renormalization programs in conjunction with our box renormalization scheme, should eliminate all of the infinities, as might be apparent from the above equation. However, proper regularization schemes should still be implemented and proper care taken when handling these infinite expressions. All the summations appearing in the eq. (3.11) are separately infinite. We want to use the Abel-Plana formula to

convert these sums into integrals. However these sums do not satisfy the stringent requirements stated in the Abel-Plana theorem for such a conversion [36, 37]. However our box renormalization scheme provides a solution by subtracting these double sums as indicated in eq. (2.2). Now we can expect this new summand to satisfy the requirements for the Abel-Plana theorem. Then all the infinities actually cancel and the result for the two-loop correction reduces to (see appendix for more details):

$$E_a^{(1)} - E_b^{(1)} = \frac{-\lambda L^2}{128\pi^2} \left[ f(a) - f(b) + 2f\left(\frac{L-a}{2}\right) - 2f\left(\frac{L-b}{2}\right) \right. \\ \left. + \frac{2m^2}{\pi} \underbrace{\left( B(a) - B(b) + 2B\left(\frac{L-a}{2}\right) - 2B\left(\frac{L-b}{2}\right) \right)}_{= \mathcal{R}} \int_0^\infty ds \sqrt{1+s^2} \right], \quad (3.12)$$

where  $f(a) = B(a) \left( \frac{B(a)}{a} - \frac{m}{a} \right)$  and  $B(a)$ , defined by the following expression

$$B(a) = -\frac{2am^2}{\pi} \int_1^\infty \frac{\sqrt{t^2-1}}{e^{2amt}-1} dt = -\frac{m}{\pi} \sum_{j=1}^\infty \frac{K_1(2amj)}{j}, \quad (3.13)$$

refers to the so called branch-cut term of the Abel-Plana summation formula and is a finite quantity. Note that the last integral in eq. (3.12) seems to diverge so it must be properly regularized. We prefer to use a regularization scheme for this integral term which is analogous to the zeta function regularization for the sums. That is, we set the power of the integrand to  $\frac{d}{2} - 1$  and in the final stage we let  $d$  approach three. Henceforth we shall also refer to this as zeta function regularization. We should note that this regularization is not sufficient to avoid the use of analytic continuation. However, as we shall see, by adding an auxiliary cutoff regulator we can ultimately avoid resorting to any analytic continuation. We thus obtain

$$\int_0^\infty (1+\kappa^2)^{\frac{d}{2}-1} d\kappa \quad \rightarrow \quad \int_0^K (1+\kappa^2)^{\frac{d}{2}-1} d\kappa = K {}_2F_1\left(\frac{1}{2}, 1 - \frac{d}{2}, \frac{3}{2}, -K^2\right). \quad (3.14)$$

The asymptotic behavior of the hypergeometric function  ${}_2F_1\left(\frac{1}{2}, 1 - \frac{d}{2}, \frac{3}{2}, -K^2\right)$  multiplied by  $K$  for the large  $K$  is,

$$K {}_2F_1\left(\frac{1}{2}, 1 - \frac{d}{2}, \frac{3}{2}, -K^2\right) \xrightarrow{K \rightarrow \infty} \frac{\sqrt{\pi}\Gamma\left(\frac{1-d}{2}\right)}{2\Gamma\left(1 - \frac{d}{2}\right)} + \frac{1}{d-1} K^{d-1} + \frac{d-2}{2(d-3)} K^{d-3} + \dots \\ \xrightarrow{d \rightarrow 3} \frac{1}{2} [K^2 + \ln(K) + \ln(2) + 1/2]. \quad (3.15)$$

Referring to eq. (3.12) and adhering to the usual prescription described for eq. (3.10), we can cancel the quadratic and logarithmic divergences using our full freedom to choose four different cutoffs in the four distinct integration regions corresponding to  $\{a_1, a_2, b_1, b_2\}$ .<sup>1</sup> Hence we get,

$$\mathcal{R} = \frac{m^2(\ln 2 + 1/2)}{\pi} \left( B(a) - B(b) + 2B\left(\frac{L-a}{2}\right) - 2B\left(\frac{L-b}{2}\right) \right). \quad (3.16)$$

---

<sup>1</sup>One may argue that ambiguities always exist in problems where one has to subtract infinite quantities, and the Casimir problems certainly fall into this category. Two methods are in common use: First is the

Using eqs. (2.2), (3.12), (3.13) and (3.16), we obtain

$$E_{\text{Cas.}}^{(1)} = \frac{-\lambda L^2 m}{128\pi^3} \sum_{j=1}^{\infty} \frac{K_1(2amj)}{j} \left[ \frac{m}{\pi a} \sum_{j'=1}^{\infty} \frac{K_1(2amj')}{j'} + \frac{m}{a} - \frac{m^2}{\pi} (\ln 2 + 1/2) \right]. \quad (3.17)$$

This is the two-loop radiative correction to the Casimir energy. This result is obviously finite, and we believe it could not have been obtained with any regularization or analytic continuation schemes in common use. Our result defers from [15, 21].

Two particular limits are interesting to calculate; The large mass limit,  $ma \gg 1$ , and small mass limit,  $m \rightarrow 0$ . In these limits eq. (3.17) becomes

$$\begin{cases} E_{\text{Cas.}}^{(1)} \xrightarrow{am \gg 1} \frac{\lambda L^2 m^2 \ln 2 + 1/2}{256\pi^3} \frac{\sqrt{m}}{\sqrt{\pi}} \sqrt{\frac{m}{a}} e^{-2am} \\ E_{\text{Cas.}}^{(1)} \xrightarrow{m \rightarrow 0} -\frac{\lambda L^2}{512\pi^4 a^3} \left( \sum_{j=1}^{\infty} \frac{1}{j^2} \right)^2 = -\frac{\lambda L^2}{18432a^3}. \end{cases} \quad (3.18)$$

Our massless limit differs from the analogous results that can be extracted from refs. [15, 21, 32, 33] by a minus sign. Figure 2 illustrates the graphs for the leading and the NLO terms for the Casimir energy as a function of  $a$ , in the massive case and its massless limit.

### 3.3 The massless case

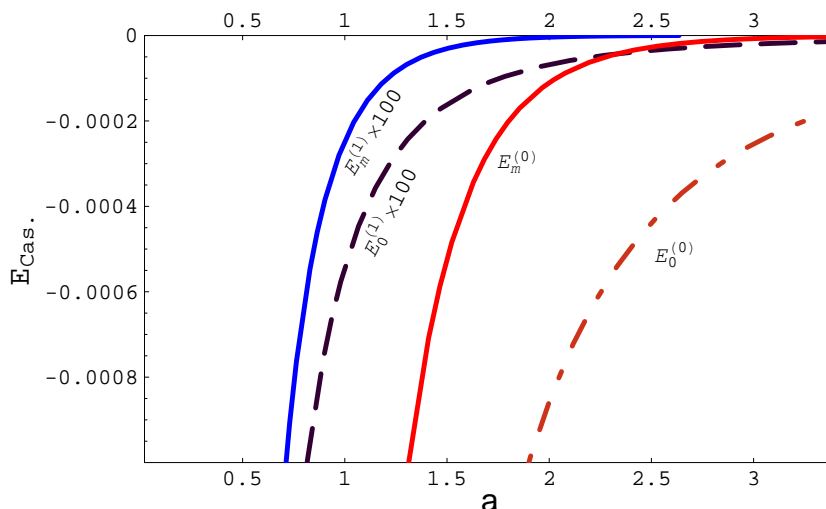
In the massless case it is sufficient to set the pole of the propagator to zero, i.e. one can set  $m = 0$  in the eq. (3.8), hence in the eq. (3.11) we have, for example  $\omega'_{a_1, n} = k_{a_1, n}$ . Accordingly this equation becomes

$$E_a^{(1)} - E_b^{(1)} = \frac{-\lambda L^2}{128} \sum_n \left[ \sum_{n'} \left( \frac{1}{a^3} n n' - \frac{1}{b^3} n n' + \frac{16}{(L-a)^3} n n' - \frac{16}{(L-b)^3} n n' \right) + \frac{1}{2a^3} n^2 - \frac{1}{2b^3} n^2 + \frac{8}{(L-a)^3} n^2 - \frac{8}{(L-b)^3} n^2 \right]. \quad (3.19)$$

We use Abel-Plana summation formula to transform the sums into integrals, then one can remove the divergent integrals by changing variables as explained before. Then the result

---

analytic continuation techniques which, although usually yield correct results, do not have a very solid physical justification and also sometimes yield infinite results. We have devoted section 4 to this subject. Second is the regularization schemes, which is what we have used. In the latter category when the problem is regularized, one can make a systematic expansion of the quantities in question in terms of the regulators. Then the terms which tend to infinity when the regulators are removed and the finite terms naturally appear separately. See for example eqs. (3.10), (3.15). What is almost invariably done is to adjust the regulators so that the singular terms exactly cancel each other, i.e. without extracting any extra finite piece from the difference between the infinite quantities (see for example [14]). This is also apparent in the leading term for the Casimir energy in eq. (2.9) where, as explained in the appendix, The four changes of variables are equivalent to choosing four different cutoffs. One could have adjusted them so that as usual the infinities cancel, but any finite term would remain. However, the well known answer is obtained only when there is no remaining extra finite term in this subtraction scheme. This is the prescription that we have used. However, we do believe that this is a subject that needs further study.



**Figure 2:** The leading terms for the Casimir energy and its NLO corrections as a function of  $a$  for the massive ( $m = 1$ ) and massless cases for  $\lambda = 0.1$ . The correction term for the massless case is always negative. However the correction term for the massive case has a peak of magnitude  $\simeq +1.57 \times 10^{-9}$  at  $a \simeq 3.04$  and asymptotes to zero from above.

is

$$E_a^{(1)} - E_b^{(1)} = -\frac{\lambda L^2}{18432} \left( \frac{1}{a^3} - \frac{1}{b^3} + \frac{16}{(L-a)^3} - \frac{16}{(L-b)^3} \right). \quad (3.20)$$

As before the eq. (2.2) gives the Casimir energy,

$$E_{\text{Cas.}}^{(1)} = -\frac{\lambda L^2}{18432 a^3}. \quad (3.21)$$

This result is in exact agreement with the small mass limit calculated in previous subsection.

Note that we are explicitly assuming that  $\delta_{m=0}(x) \neq 0$ . We like to stress that this quantity should not in general be a priori set to zero. This is in contrast to the view expressed in for example refs. [32–35]. This is yet another important counterexample for the validity of analytic continuation: As is well known the massless limit of the analytic continuation of the mass counterterm in  $\phi^4$  theory is zero for space-time dimensions bigger than two. However one cannot renormalize the massless theory without the mass counterterm (see for example [23]).

#### 4. On the analytic continuation techniques

In the previous section we saw that enclosing the whole system in a box is very useful for removing all of the infinities which appear in the calculations. In this section, for comparison purposes, we repeat some of those steps using analytic continuation techniques. We can then show or point out three examples and two counterexamples for the validity of the analytic continuation techniques.

The first example for the validity of these techniques is the derivation of the leading term mentioned in the section 2. The second example is the following: Starting with eq. (3.8) which is obviously infinite, one can regularize it by dimensional regularization:

$$G_{a_1}(x, x) = \frac{2\Gamma(1 - d/2)}{(4\pi)^{d/2}a} \sum_n \omega'_{a_1, n}{}^{d-2} \sin^2 \left[ k_{a_1, n} \left( z + \frac{a}{2} \right) \right]. \quad (4.1)$$

This equality holds only for  $d < 1$ , however it can be analytically continued to  $d \geq 1$ . Putting this Green's function in eq. (3.7) and performing the space integration and repeating analogous steps for the other regions, one easily obtains:

$$E_a^{(1)} - E_b^{(1)} = \frac{\lambda L^{d-1} \Gamma^2(1 - d/2)}{8(4\pi)^d} \sum_n \left[ \sum_{n'} \left( \frac{\omega'_{a_1, n}{}^{d-2} \omega'_{a_1, n'}{}^{d-2}}{a} - \frac{\omega'_{b_1, n}{}^{d-2} \omega'_{b_1, n'}{}^{d-2}}{b} \right. \right. \quad (4.2)$$

$$\left. \left. + 4 \frac{\omega'_{a_2, n}{}^{d-2} \omega'_{a_2, n'}{}^{d-2}}{L - a} - 4 \frac{\omega'_{b_2, n}{}^{d-2} \omega'_{b_2, n'}{}^{d-2}}{L - b} \right) \right. \\ \left. + \frac{1}{2} \left( \frac{\omega'_{a_1, n}{}^{2d-4}}{a} - \frac{\omega'_{b_1, n}{}^{2d-4}}{b} + 4 \frac{\omega'_{a_2, n}{}^{2d-4}}{L - a} - 4 \frac{\omega'_{b_2, n}{}^{2d-4}}{L - b} \right) \right].$$

Putting  $d = 3$  in eq. (4.2), eq. (3.11) is obtained. Consequently we have shown that the analytic continuation for the Green's function, in this case, gives the correct result. The third example for the validity of the analytic continuation technique is that with its mere use one can obtain eq. (3.20) from eq. (3.19).

The first counterexample can be illustrated in the continuation of derivation of the NLO Casimir energy from eq. (4.2). Using the Abel-Plana formula we encounter the following term

$$\int_0^\infty (1 + \kappa^2)^{\frac{d}{2}-1} d\kappa = \frac{\sqrt{\pi} \Gamma(\frac{1-d}{2})}{2\Gamma(1 - \frac{d}{2})}, \quad (4.3)$$

and this leads to the a term which contains  $\Gamma(1 - d/2)\Gamma(\frac{1-d}{2})$ . This term is infinite in any integer space dimension. This is an obvious counterexample for the use of this analytic continuation technique. As for the second counterexample, it concerns the massless limit of the analytic continuation of the mass counterterm, as explained at the end of the previous subsection.

## 5. Conclusions

We have introduced a new concept in this paper. We have insisted that the renormalization program should completely and self-consistently take into account the boundary conditions or any possible nontrivial backgrounds which break the translational invariance of the system. We have shown that the problem is self-contained and the above program is accomplishable. To be more specific, there should be no need to import counterterms from the free theory, or even supplementing them with the *ad hoc* attachment of extra surface terms, to remedy the divergences inherent in this theory. In general this breaking

of the translational invariance is reflected in the nontrivial position dependence of all the  $n$ -point functions. As we have shown, this has profound consequences. For example in the case of renormalized perturbation theory, the counterterms and hence the radiative corrections to parameters of the theory, i.e.  $m$  and  $\lambda$ , automatically turn out to be position dependent. In this regard we disagree with the authors who use the free counterterms (see the Introduction for actual references). Obviously we still need a few experimental input parameters for the complete renormalization program, such as  $m(x)$  for some  $x$ . However, the interesting point is that the theory then completely determines  $m(x)$ .

Secondly we have used a supplementary renormalization scheme to be used along side the usual renormalization program. In computations of these sorts, there usually appears infinities which can sometimes be removed by the usual renormalization programs that often contain some sort of analytic continuation. These procedures are sometimes ambiguous. Our scheme is simply to confine the whole physical system in a box, and to compute the difference between the values of the physical quantity in question in two different configurations. Use of this procedure removes all of the ambiguities associated with the appearance of the infinities, and we use the usual prescription for removing the infinities in the regulated theory. Using our method, we have computed the zero and first order radiative correction to the Casimir energy for the massive and massless real scalar field in 3+1 dimensions. For the zero order, our results are identical with what exists in the literature. However, our first order results are markedly different from those reported in refs. [15, 21]. Our results for the massive and massless cases are different from theirs due to the aforementioned conceptual differences. As we have shown our results for the massless case and the massless limit of the massive case are identical. However, the massless limit of their massive results agrees with the “exact” results obtained in [32, 33], who set  $\delta_m$  equal to zero in their massless cases, or their equivalent. This is our second main difference in approach to the problem. As mentioned before, we believe that  $\delta_m$  should not be arbitrarily set to zero even in the massless case, since in that case the renormalization conditions can no longer be fully implemented, although the theory is still in principle renormalizable. In this regards, we like to emphasize that their results for massless case is infinite in 1+1 dimensions.

## Acknowledgments

It is a great pleasure for us to acknowledge the useful comments of Michael E. Peskin which helped us resolve one of the main difficulties in our calculations. We would also like to thank the Referee for illuminating comments. This research was supported by the office of research of the Shahid Beheshti University. R.M. would also like to thank the Qom University for financial support.

## A. Use of the Abel-Plana summation formula in our subtraction scheme

In this appendix we present the details of the calculations leading to eq. (3.12). The

Abel-Plana summation formula (see for example [37]) is:

$$\sum_{n=1}^{\infty} f(n) = -\frac{f(0)}{2} + \int_0^{\infty} f(x)dx + i \int_0^{\infty} \frac{f(it) - f(-it)}{e^{2\pi t} - 1} dt. \quad (\text{A.1})$$

In order to obtain first-order radiative correction, just as we discussed in section 2 for leading term, we need to compute  $E_a^{(1)} - E_b^{(1)}$ . We rewrite the eq. (3.11) as,

$$\begin{aligned} E_a^{(1)} - E_b^{(1)} = & \frac{-\lambda L^2}{128\pi^2} \sum_n \left[ \sum_{n'} \left( \frac{1}{a} S(a, n) S_{a_1}(n') - \frac{1}{b} S(b, n) S(b, n') \right) \right. \\ & + \frac{4}{L-a} S\left(\frac{L-a}{2}, n\right) S\left(\frac{L-a}{2}, n'\right) - \frac{4}{L-b} S\left(\frac{L-b}{2}, n\right) S\left(\frac{L-b}{2}, n'\right) \\ & \left. + \frac{S^2(a, n)}{2a} - \frac{S^2(b, n)}{2b} + \frac{2S^2(\frac{L-a}{2}, n)}{L-a} - \frac{2S^2(\frac{L-b}{2}, n)}{L-b} \right], \end{aligned} \quad (\text{A.2})$$

where  $S(a, n) = \left(m^2 + \frac{n^2\pi^2}{a^2}\right)^{1/2}$ . Using the Abel-Plana formula eq. (A.1), we obtain

$$\begin{aligned} E_a^{(1)} - E_b^{(1)} = & \frac{-\lambda L^2}{128\pi^2} \sum_n \left( \frac{m}{a} S(a, n) - \frac{m}{b} S(b, n) + \frac{4m}{L-a} S\left(\frac{L-a}{2}, n\right) - \frac{4m}{L-b} S\left(\frac{L-b}{2}, n\right) \right) \\ & + \frac{1}{\pi} \int_0^{\infty} \sqrt{m^2 + s'^2} ds' \left[ S(a, n) - S(b, n) + 2S\left(\frac{L-a}{2}, n\right) - 2S\left(\frac{L-b}{2}, n\right) \right] \\ & + \frac{B(a)S(a, n)}{a} - \frac{B(b)S(a, n)}{b} + \frac{4B(\frac{L-a}{2})S(a, n)}{L-a} - \frac{4B(\frac{L-b}{2})S(a, n)}{L-b} \\ & + \frac{S^2(a, n)}{2a} - \frac{S^2(b, n)}{2b} + \frac{2S^2(\frac{L-a}{2}, n)}{L-a} - \frac{2S^2(\frac{L-b}{2}, n)}{L-b} \Big), \end{aligned} \quad (\text{A.3})$$

Using eq. (A.1) again, and making appropriate changes of variables to make the integrals dimensionless, all the actual infinities cancel and we finally obtain,

$$\begin{aligned} E_a^{(1)} - E_b^{(1)} = & \frac{-\lambda L^2}{128\pi^2} \left[ f(a) - f(b) + 2f\left(\frac{L-a}{2}\right) - 2f\left(\frac{L-b}{2}\right) \right] \\ & + \frac{2m^2}{\pi} \left( B(a) - B(b) + 2B\left(\frac{L-a}{2}\right) - 2B\left(\frac{L-b}{2}\right) \right) \int_0^{\infty} ds \sqrt{1+s^2} \Big] \end{aligned} \quad (\text{A.4})$$

It is important to note that all these cancelations are easily accomplished using our supplementary box renormalization scheme. On a minor note, it is interesting to note that the changes of the variables leading to the cancelation of infinities are, surprisingly, equivalent to setting different cutoff regularizations on the upper limits of the integrals. Equation (A.4) is our main equation for the NLO Casimir energy, and appears in the text as eq. (3.12), and is analyzed further there.

## References

- [1] H.B.G. Casimir, *On the attraction between two perfectly conducting plates*, *Proc. Kon. Nederl. Akad. Wet.* **51** (1948) 793.



- [2] H.B.G. Casimir and D. Polder, *The influence of retardation on the London-van der Waals forces*, *Phys. Rev.* **73** (1948) 360.
- [3] J. Ambjørn and S. Wolfram, *Properties of the vacuum, 1. Mechanical and thermodynamic*, *Ann. Phys. (NY)* **147** (1983) 1.
- [4] N.F. Svaiter and B.F. Svaiter, *Casimir effect in a d-dimensional flat space-time and the cutoff method*, *J. Math. Phys.* **32** (1991) 175;  
K.A. Milton, *The Casimir effect*, World Scientific (2001); *The Casimir effect: recent controversies and progress*, *J. Phys.* **A 37** (2004) 209.
- [5] M. Bordag, D. Robaschik and E. Wieczorek, *Quantum field theoretic treatment of the Casimir effect* *Ann. Phys. (NY)* **165** (1985) 192;  
M. Bordag and J. Lindig, *Radiative correction to the casimir force on a sphere*, *Phys. Rev. D* **58** (1998) 045003 [[hep-th/9801129](#)];  
D. Robaschik, K. Scharnhorst and E. Wieczorek, *Radiative corrections to the Casimir pressure under the influence of temperature and external fields*, *Ann. Phys. (NY)* **174** (1987) 401;  
M. Bordag and K. Scharnhorst, *O(alpha) radiative correction to the Casimir energy for penetrable mirrors*, *Phys. Rev. Lett.* **81** (1998) 3815 [[hep-th/9807121](#)];  
S.-S. Xue, *Casimir effect of scalar field on S(n - 1) manifold*, *Commun. Theor. Phys.* **11** (1989) 243.
- [6] F. Ravndal and J.B. Thomassen, *Radiative corrections to the Casimir energy and effective field theory*, *Phys. Rev. D* **63** (2001) 113007 [[hep-th/0101131](#)].
- [7] X. Kong and F. Ravndal, *Radiative corrections to the Casimir energy*, *Phys. Rev. Lett.* **79** (1997) 545 [[quant-ph/9703019](#)].
- [8] K. Melnikov, *Radiative corrections to the Casimir force and effective field theories*, *Phys. Rev. D* **64** (2001) 045002 [[hep-ph/0101228](#)].
- [9] L.H. Ford, *Casimir effect for a self-interacting scalar field*, *Proc. Roy. Soc. London A* **368** (1979) 305.
- [10] B.S. Kay, *The Casimir effect without magic*, *Phys. Rev. D* **20** (1979) 3052.
- [11] D.J. Toms, *Symmetry breaking and mass generation by space-time topology*, *Phys. Rev. D* **21** (1980) 2805.
- [12] K. Langfeld, F. Schmuser and H. Reinhardt, *Casimir effect of strongly interacting scalar fields*, *Phys. Rev. D* **51** (1995) 765 [[hep-ph/9307258](#)].
- [13] L.C. de Albuquerque, *Casimir pressure at two loops and soft boundaries at finite temperature*, *Phys. Rev. D* **55** (1997) 7754.
- [14] M. Bordag, U. Mohideen and V.M. Mostepanenko, *New developments in the Casimir effect*, *Phys. Rept.* **353** (2001) 1 [[quant-ph/0106045](#)].
- [15] F.A. Baron, R.M. Cavalcanti and C. Farina, *Radiative corrections to the Casimir effect for the massive scalar field*, *Nucl. Phys.* **127** (Proc. Suppl.) (2004) 118.
- [16] F.A. Barone, R.M. Cavalcanti and C. Farina, *Radiative corrections to the Casimir energy in the  $\lambda|\phi|^4$  model under quasi-periodic boundary conditions*, [hep-th/0312169](#).
- [17] N. Graham, R.L. Jaffe and H. Weigel, *Casimir effects in renormalizable quantum field theories*, *Int. J. Mod. Phys. A* **17** (2002) 846 [[hep-th/0201148](#)].

- [18] N. Graham et al., *Calculating vacuum energies in renormalizable quantum field theories: a new approach to the Casimir problem*, *Nucl. Phys. B* **645** (2002) 49 [[hep-th/0207120](#)]; *Casimir energies in light of quantum field theory*, *Phys. Lett. B* **572** (2003) 196 [[hep-th/0207205](#)];  
N. Graham et al., *The Dirichlet Casimir problem*, *Nucl. Phys. B* **677** (2004) 379 [[hep-th/0309130](#)].
- [19] C.D. Fosco and N.F. Svaiter, *Finite size effects in the anisotropic  $\lambda/4!(\phi_1^4 + \phi_2^4)$  model*, *J. Math. Phys.* **42** (2001) 5185 [[hep-th/9910068](#)].
- [20] J.A. Nogueira and P.L. Barbieri, *Boundary conditions as mass generation mechanism for real scalar fields*, *Braz. J. Phys.* **32** (2002) 798;  
M.I. Caicedo and N.F. Svaiter, *Effective lagrangians for scalar fields and finite size effects in field theory*, *J. Math. Phys.* **45** (2004) 179 [[hep-th/0207202](#)];  
N.F. Svaiter, *Finite size effects in thermal field theory*, *J. Math. Phys.* **45** (2004) 4524 [[hep-th/0410016](#)];  
M.A. Alcalde, G.F. Hidalgo and N.F. Svaiter, *The two-loop massless  $(\lambda/4!)\phi^4$  model in nontranslational invariant domain*, *J. Math. Phys.* **47** (2006) 052303.
- [21] R.M. Cavalcanti, C. Farina and F.A. Barone, *Radiative corrections to casimir effect in the  $\lambda\phi^4$  model*, [hep-th/0604200](#).
- [22] A.J. Buras, *Asymptotic freedom in deep inelastic processes in the leading order and beyond*, *Rev. Mod. Phys.* **52** (1980) 199.
- [23] M.E. Peskin and D.V. Schroeder, *An introduction to quantum field theory*, Addison-Wesley (1995).
- [24] H.J. de Vega, *Two-loop quantum corrections to the soliton mass in two-dimensional scalar field theories*, *Nucl. Phys. B* **115** (1976) 411;  
M.A. Lohe and D.M. O'Brien, *Soliton mass corrections and explicit models in two-dimensions*, *Phys. Rev. D* **23** (1981) 1771;  
N. Graham and R.L. Jaffe, *Fermionic one-loop corrections to soliton energies in 1 + 1 dimensions*, *Nucl. Phys. B* **549** (1999) 516 [[hep-th/9901023](#)];  
E. Farhi, N. Graham, R.L. Jaffe and H. Weigel, *Heavy fermion stabilization of solitons in 1 + 1 dimensions*, *Nucl. Phys. B* **585** (2000) 443 [[hep-th/0003144](#)];  
A. Alonso Izquierdo, W. Garcia Fuertes, M.A. Gonzalez León and J. Mateos Guilarte, *Semi-classical mass of quantum K-component topological kinks*, *Nucl. Phys. B* **638** (2002) 378 [[hep-th/0205137](#)];  
A.A. Izquierdo, W.G. Fuertes, M.A. González León and J.M. Guilarte, *Generalized zeta functions and one-loop corrections to quantum kink masses*, *Nucl. Phys. B* **635** (2002) 525;  
A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *The anomaly in the central charge of the supersymmetric kink from dimensional regularization and reduction*, *Nucl. Phys. B* **648** (2003) 174 [[hep-th/0207051](#)];  
A.A. Izquierdo, W.G. Fuertes, M.A. González León and J.M. Guilarte, *One-loop correction to classical masses of quantum kink families*, *Nucl. Phys. B* **681** (2004) 163 [[hep-th/0304125](#)].
- [25] A. Rebhan and P. van Nieuwenhuizen, *No saturation of the quantum bogomolnyi bound by two-dimensional  $N = 1$  supersymmetric solitons*, *Nucl. Phys. B* **508** (1997) 449 [[hep-th/9707163](#)];  
H. Nastase, M.A. Stephanov, P. van Nieuwenhuizen and A. Rebhan, *Topological boundary conditions, the BPS bound and elimination of ambiguities in the quantum mass of solitons*, *Nucl. Phys. B* **542** (1999) 471 [[hep-th/9802074](#)];

- A.S. Goldhaber, A. Litvintsev and P. van Nieuwenhuizen, *Mode regularization of the SUSY sphaleron and kink: zero modes and discrete gauge symmetry*, *Phys. Rev. D* **64** (2001) 045013 [[hep-th/0011258](#)];
- G. Flores-Hidalgo, *One loop renormalization of soliton quantum mass corrections in 1 + 1 dimensional scalar field theory models*, *Phys. Lett. B* **542** (2002) 282 [[hep-th/0206047](#)];
- A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *New developments in the quantization of supersymmetric solitons (kinks, vortices and monopoles)*, *Braz. J. Phys.* **34** (2004) 1273;
- A.S. Goldhaber, A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, *Quantum corrections to mass and central charge of supersymmetric solitons*, *Phys. Rept.* **398** (2004) 179 [[hep-th/0401152](#)].
- [26] R.F. Dashen, B. Hasslacher and A. Neveu, *Nonperturbative methods and extended hadron models in field theory. 1. Semiclassical functional methods*, *Phys. Rev. D* **10** (1974) 4114; *Nonperturbative methods and extended hadron models in field theory. 2. Two-dimensional models and extended hadrons*, *Phys. Rev. D* **10** (1974) 4130; *Semiclassical bound states in an asymptotically free theory*, *Phys. Rev. D* **12** (1975) 2443.
- [27] R. Moazzemi and S.S. Gousheh, *A new renormalization approach to the Dirichlet Casimir effect for  $\phi^4$  theory in (1+1) dimensions*, [arXiv:0708.3428](#), to appear in *Phys. Lett. B*.
- [28] Work in progress.
- [29] S. Blau, M. Visser and A. Wipf, *Zeta functions and the Casimir energy*, *Nucl. Phys. B* **310** (1988) 163;  
C.M. Bender and K.A. Milton, *Casimir effect for a D-dimensional sphere*, *Phys. Rev. D* **50** (1994) 6547 [[hep-th/9406048](#)].
- [30] T.H. Boyer, *Quantum electromagnetic zero-point energy of a conducting spherical shell and the Casimir model for a charged particle*, *Phys. Rev.* **174** (1968) 1764.
- [31] V.V. Nesterenko, *Boundary conditions at spatial infinity for fields in Casimir calculations*, *J. Phys. A* **39** (2006) 6609.
- [32] K. Symanzik, *Schrödinger representation and Casimir effect in renormalizable quantum field theory*, *Nucl. Phys. B* **190** (1981) 1.
- [33] M. Krech and S. Dietrich, *Free energy and specific heat of critical films and surfaces*, *Phys. Rev. A* **46** (1992) 1886.
- [34] T. van Ritbergen and R.G. Stuart, *Hadronic contributions to the muon lifetime*, *Phys. Lett. B* **437** (1998) 201 [[hep-ph/9802341](#)].
- [35] G. von Gersdorff and A. Hebecker, *Radius stabilization by two-loop Casimir energy*, *Nucl. Phys. B* **720** (2005) 211.
- [36] P. Henrici, *Applied and computational complex analysis. Vol. 1*, Wiley, New York (1984);  
E.T. Whittaker and G.N. Watson, *A course of modern analysis*, Cambridge University Press (1958).
- [37] A.A. Saharian, *The generalized Abel-Plana formula: applications to Bessel functions and Casimir effect*, [hep-th/0002239](#); *The generalized Abel-Plana formula with applications to Bessel functions and Casimir effect*, [arXiv:0708.1187](#).